Existence and Approximation of Solutions for Vector Mixed Variational Inequalities in Reflexive Banach Space

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Introduction

- Variational Inequalities is an inequality that include an operator which is solved for all possible values of given variable over a convex set.
- Classical Variational Inequalities started with pioneering work of Stampacchia in 1960.
- It is an inequality of finding $u \in K \subseteq \mathbb{R}^n$ such that

$$(F(u), v-u) \geq 0 \quad \forall v \in K$$

where $F : K \to \mathbb{R}^n$
- It includes plethora of problems like Optimization, Fixed point theory, Economics etc.
- Mixed Variational Inequalities was introduced by Lescarret and Browder in 1966 and defined as find $u \in K$ such that $(F(u), v-u) + f(v) - f(u) \geq 0 \quad \forall v \in K$ where $f : K \to \mathbb{R}$ is convex function.

Example: Let $f : I = [a, b] \to \mathbb{R}$ be smooth function such that $f(u) = \min_{v \in I} f(v)$ then its variational formulation is

$$f'(u)(v-u) \geq 0 \ \forall v \in I$$

Objective

To establish the existence and uniqueness of solution for vector mixed variational inequality.

Methods

- Iterative approach
- Optimization approach
- Topological approach

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Notations and Definitions

Let X be a reflexive Banach space and Y be a real Banach space. Let $K \subset X$ be nonempty, closed and convex set, and $C \subset Y$ be a closed, convex and pointed cone with apex at the origin. The dual of Y is denoted by Y^*. We denote by C^* the positive polar cone of C,

i.e $\quad C^* = \{u \in B : \langle u, v\rangle \geq 0, \ \ \forall v \in C\}$ and $intC^* \subset \{u \in B : \langle u, v\rangle > 0, \ \ \forall v \in C\}$ Then C induces a vector ordering in B as follows:

$$u \leq v \text{ if and only if } v-u \in C,$$
$$u \nleq v \text{ if and only if } v-u \notin C,$$
$$u < v \text{ if and only if } v-u \in intC,$$
$$u \nless v \text{ if and only if } v-u \notin intC.$$

Then $(B, \leq)$ is a partial ordering.

- A mapping $G : K \to L(X, Y)$ is said to be (η, f)-C- pseudomonotone if for all $u, v \in K$, we have
$\langle G(u), \eta(v, u)\rangle + f(v) - f(u) \notin -intC$
$\implies \langle G(v), \eta(v, u)\rangle + f(v) - f(u) \notin -intC.$
- Let $K \subset X$ be a convex set. A mapping $f : K \to Y$ is said to be C- convex if for all $u, v \in K$ and for each $t \in [0, 1]$, we have
$f(tu + (1-t)v) - tf(u) - (1-t)f(v) \in -C$

Formulation of the problem

Let K be a nonempty, closed and convex subset of a reflexive Banach space and B be a real Banach space. Let $G : K \to L(E, B)$ be an operator, $f : K \to B$ be a C- convex function and $\eta : K \times K \to E$ be a mapping

- Vector Mixed Varitional inequality (VMVI 1) is defined as
find $u \in K$ such that
$\langle G(u), \eta(v, u)\rangle + f(v) - f(u) \notin -intC, \quad \forall v \in K$
- Another Vector Mixed Varitional inequality (VMVI 2) is defined as
find $u \in K$ such that
$\langle G(v), \eta(v, u)\rangle + f(v) - f(u) \notin -intC, \quad \forall v \in K$

Results

Theorem 1 *Let K be a nonempty closed and convex subset of a real reflexive Banach space E and B be a real Banach space ordered by a closed, convex and pointed cone C. Let $G : K \to L(E, B)$ is (η, f)-C-pseudomonotone and η-C-hemicontinuous. Let the map $u \mapsto \langle G(w), \eta(u, v)\rangle$ is C-convex. Suppose the mapping $f : K \to B$ is C-convex and $\eta(u, u) = 0$ for all $u \in K$. Then the solution set of VMVI 1 and VMVI 2 coincide.*

Theorem 2 *Let K be a nonempty closed, convex and unbounded subset of a real reflexive Banach space E and B be a real Banach spaces ordered by a closed, convex and pointed cone C. Let $G : K \to L(E, B)$ be an η-C-hemicontinuous mapping and $f : K \to B$ be a C-convex function. Suppose that the following conditions hold:*

1. *G is (η, f)-C-pseudomonotone;*
2. *the map $u \mapsto \langle G(w), \eta(u, v)\rangle$ is C-convex;*
3. *$u \mapsto \langle G(w), \eta(v, u)\rangle$ is C-upper semicontinuous;*
4. *f is C-lower semicontinuous;*
5. *$\eta(u, u) = 0$;*
6. *G is weakly coercive with respect to f that is there exists $v_0 \in K$ such that $\langle G(u), \eta(v_0, u)\rangle + f(v_0) - f(u) \in -intC$, whenever $||u|| \to +\infty$ and $u \in K$.*

Then $VMVI1$ has a solution.

Auxiliary Problem : For a given $u \in K$ and a real $\theta > 0$, find a $w \in K$ such that

$$\theta\left[\langle G(w), \eta(v, w)\rangle + f(v) - f(w)\right] + \langle Tv - Tw, \eta(w, u)\rangle \notin -intC, \quad \forall v \in K.$$

Iterative Scheme : At nth step, for given $u = u_n$, find $w = u_{n+1}$ such that

$$\theta_{n+1}\left[\langle G(u_{n+1}), \eta(v, u_{n+1})\rangle + f(v) - f(u_{n+1})\right] + \langle Tv - Tu_{n+1}, \eta(u_{n+1}, u_n)\rangle \notin -intC.$$

Conclusion

- This Vector mixed variational inequality contains an operator which is a member of a class of operators and ordering is partial ordering on Banach space induced by a cone .
- We have investigated the existence and uniqueness of solution for VMVI 1 and established the convergence of the sequence generated through auxiliary principle technique.

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References

1. A. Arunchai et al. 2014. Existence theorems for relaxed η-α pseudomonotone and strictly η-quasimonotone generalized variational-like inequalities. 2014(1), 1-13.
2. Bijaya Kumar Sahu et al. 2020. *Existence results for mixed equilibrium problems involving set-valued operators with applications to quasi-hemivariational inequalities*. J. Optim. Theory Appl, 184(3), 810-823.
3. Bijaya Kumar Sahu et al. 2020. *Densely relaxed pseudomonotone and quasimonotone generalized variational-like inequalities*. Optimization 70(2), 413-435.
4. Chadli, O., Mohpatra, R. N 2020. *On the existence and approximation of solutions for bilevel vector variational inequalities in Banach spaces*. J. Nonlinear. Anal., 4(1), 135-151.
5. Stampachia, G., and D. Kinderlehrer 1986, *An introduction to variational inequalities and their application*. Academic Press, New York.