Symmetric Group of Julia Sets of Rational Functions

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1. Abstract

We introduce a form of rational maps, the symmetric group of those Julia set contains rotations about the origin. We prove that Konig's methods and Chebyshev-Halley's family are of this form when they are applied to a normalized polynomial. From this, few results have been deduced and displayed here.

2. Introduction

For a non-constant rational function R with degree at least two, its Fatou set is defined as $F(R) = \{z \in \hat{\mathbb{C}}: z$ has a neighbourhood where $\{R^n\}_{n>0}$ is normal$\}$. The Julia set, denoted by $J(R)$, is the complement of $F(R)$ in the extended complex plane $\hat{\mathbb{C}}$. One of the most recent works in Complex Dynamical System is to find maps that keep the Julia set invariant. Beardon classified the Symmetric group of Julia sets of polynomials and completed the study in this direction. However, this is yet to be completely understood for rational maps.

3. Symmetric Group of Julia Sets of Polynomials

Let $p(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_0,\ a_d \neq 0, d \geq 2$.

- Centroid of p : $\xi = -\frac{a_{d-1}}{d a_d}$: Average of all preimages of any point in $\mathbb{C}$.

Definition: The symmetric group of $J(p)$, denoted by $\sum(p)$, defined as the group of Euclidean symmetries γ ($\gamma(z) = az + b: |a| = 1$) such that $\gamma(J(p)) = J(p)$.
Precisely, $\sum(p)$ is some group of Euclidean rotations about ξ.

Property: For any affine map σ, $\sum(\sigma p \sigma^{-1}) = \sigma \sum(p)\, \sigma^{-1}$.

Normalization: Conjugate p with an affine map σ such that the new polynomial $\tilde{p}$ is of the form

$$\tilde{p}(z) = (\sigma p \sigma^{-1})(z) = z^d + 0.z^{d-1} + \cdots + c$$

i.e., $\tilde{p}$ is monic and centroid of it is the origin. Then $\tilde{p}$ is called a normalized polynomial and can always be written in the form

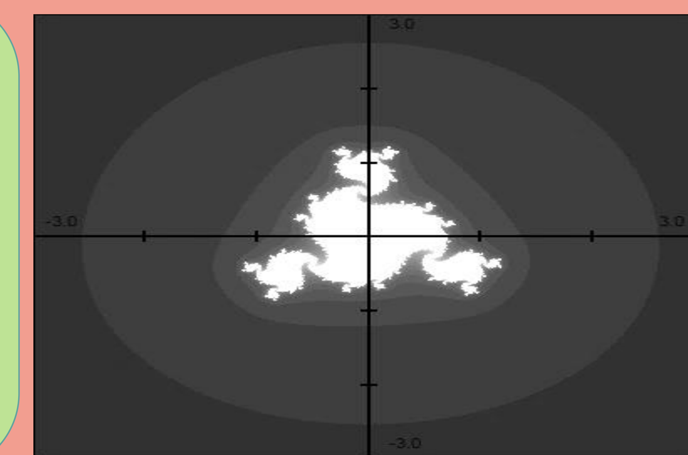
$$\tilde{p}(z) = z^a p_0(z^b) \qquad \ldots(1)$$

where a and b are maximal for this expression.
If $a = d$, then $\tilde{p}(z) = z^d$, a monomial, and hence, $J(\tilde{p})$ is a circle.

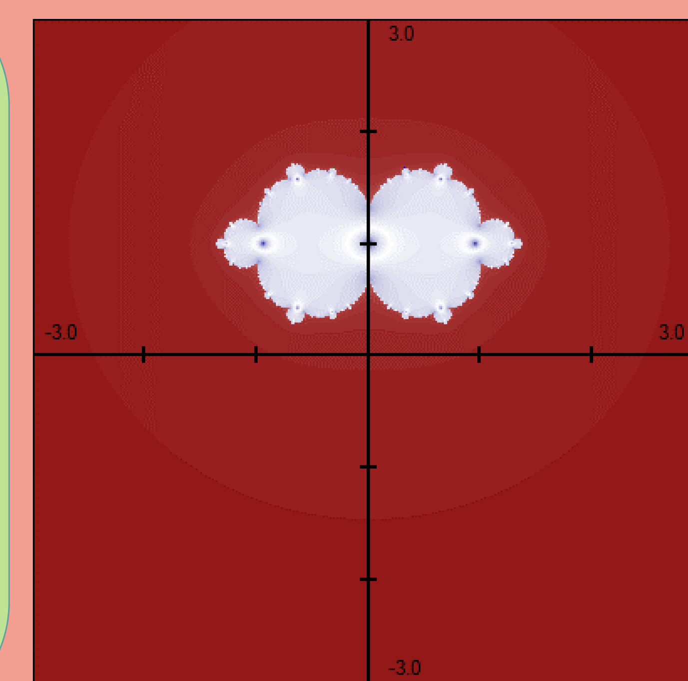
Theorem [1]: For $0 \leq a < d$, $\sum(\tilde{p}) = \{\gamma: \gamma(z) = \lambda z, where\ \lambda^b = 1\}$.

Examples

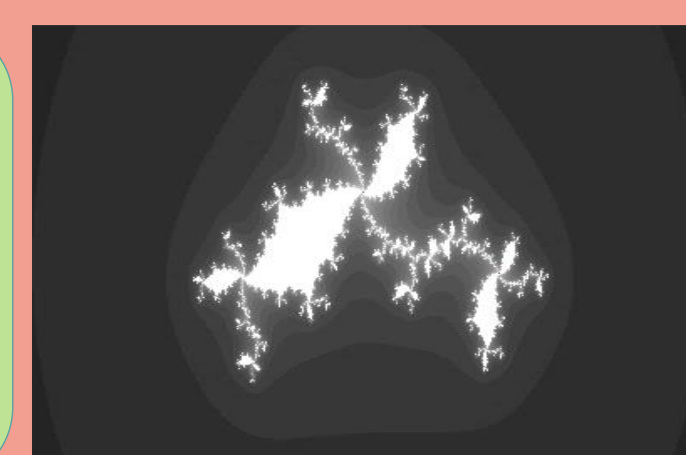
- Consider $p(z) = z^3 - 0.13 + 0.86i$. This is a normalised polynomial, can be written of the form (1), where $a = 0, b = 3$, hence, $\sum p = \{\sigma: \sigma(z) = \lambda z$ where $\lambda^3 = 1\}$.



- Consider $p(z) = z^3 - 3iz^2 - 3.9z + 2.9i$. The centroid of p is i. Then, for $\gamma(z) = z - i$, $\tilde{p}(z) = (\gamma p \gamma^{-1})(z) = z(z^2 - 0.9)$. $\tilde{p}$ is normalised and of the form (1), where $a = 1, b = 2$. Therefore, $\sum(\tilde{p}) = \{I, \alpha\}$, where $\alpha(z) = -z$, and, $\sum(p) = \{I, \beta\}$, where $\beta(z) = \gamma^{-1}\alpha\gamma(z) = -z + 2i$.



- Consider $p(z) = z^3 - z - 0.1 + 0.6i$. This is a normalised polynomial, of the form (1), where $a = 0, b = 1$, therefore, and $\sum p = \{I\}$.

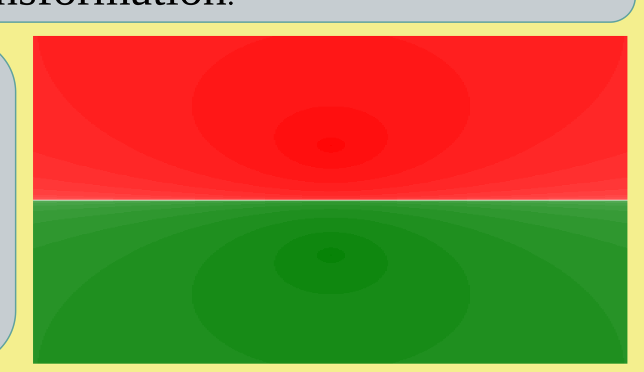


4. Results on Julia Sets of Rational Maps

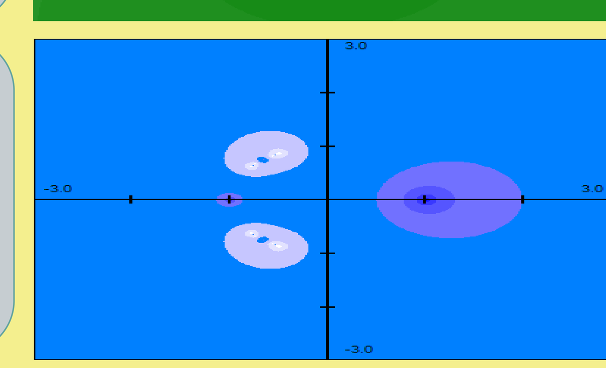
Why a new definition is necessary?

- $R(z) = 1 - \frac{2}{z^2}$, $J(R)$ is the whole complex plane $\hat{\mathbb{C}}$, hence, is invariant under every Mobius transformation.

- $R(z) = \frac{z^2-1}{2z}$
$J(R)$ is the real line, hence, is invariant under translation by any real number.



- $R(z) = \frac{2z^2+2z+1}{z^2+2z+2}$, $R(z) = \frac{1}{R(\frac{1}{z})}$, therefore, $J(R) = \gamma(J(R))$, where $\gamma(z) = \frac{1}{z}$.



Let R be a rational map, $\deg(R) \geq 2$. Then $\sum(R) = \{\varphi: \varphi$ is a Mobius map such that $\varphi(J(R)) = J(R)\}$.

Boyd's result [2]: If $\deg(R) \geq 2$ and $J(R) = J(R) + 1$ and ∞ is either periodic or preperiodic, then $J(R)$ is either $\hat{\mathbb{C}}$ or a horizontal line.

An Improvement →

Theorem: If $\deg(R) \geq 2$ and $J(R) = J(R) + a, a \in \mathbb{C}$ and ∞ is either periodic or preperiodic, then $J(R)$ is either $\hat{\mathbb{C}}$ or a line.

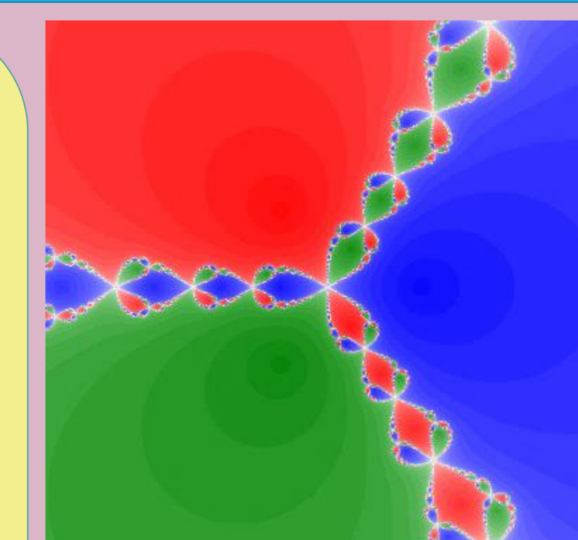
Theorem: Consider rational maps of the form

$$R(z) = z^k \left(1 + \frac{\tilde{P}(z)}{\tilde{Q}(z)}\right) \qquad \ldots(2)$$

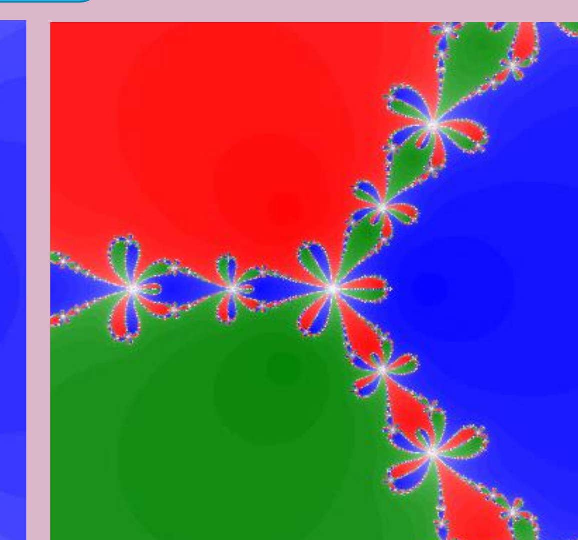
where $k \geq 1$, $\tilde{P}$ and $\tilde{Q}$ are polynomials that can be written as $\tilde{P}(z) = P(z^{b_1})$, where b_1 is maximal, and $\tilde{Q}(z) = Q(z^{b_2})$, where b_2 is maximal, then $\sum(R)$ contains $\{\sigma: \sigma(z) = \lambda z, \lambda^b = 1\}$ as a subgroup, where $b = \gcd(b_1, b_2)$.

Application on Chebyshev-Halley Family and Konig's Methods

Let F_p represents either the Konig's methods or a Chebyshev-Halley family applied to a normalised polynomial $p(z) = z^a p_0(z^b)$.
Theorem: F_p can be written in the form (2). Hence $\sum(p) \subseteq \sum(F_p)$.



The Julia set of the Newton method applied to $z^3 - 1$



The Julia set of the Chebyshev's method applied to $z^3 - 1$

The result of Liu and Gao [3]: $\sum(K_{f,n})$, $K_{f,n}$ is the Konig's method of order n applied to a polynomial f, contains $\varphi(z) = z + 1$ if and only if $f(z) = c(z-a)^k(z-b)^k$ with $c \in \mathbb{C}^*$ and $a \neq b$ but $Re(a) = Re(b)$.

Theorem (Symmetry on Chebyshev's method)
$J(C_f)$, the Julia set of Chebyshev's method applied to a polynomial f, can never be a line. Hence, $\sum(C_f)$ does not contain any translation.

5. References

[1] A. F. Beardon, Iteration of Rational Functions. Grad. Texts in Math. 132, Springer-Verlag, 1991.
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[3] G. Liu, J. Gao, Symmetries of the Julia sets of Konig's methods for polynomials, J. Math. Anal. Appl. 432 (2015) 356-366.

6. Acknowledgement

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